Acta Sci. Math., 40 (1978), 371—374

On Möbius bounded operators

ALLEN L. SHIELDS

An operator T (that is, a bounded linear transformation) on a Banach space X is said to be power bounded if $\|T^n\| \leqq M$ $(n=1, 2, \ldots)$. It is said to be Möbius bounded if $\|\varphi(T)\| \leqq c$ $(\varphi \in \mathcal{M})$. Here $\mathcal{M}$ denotes the Möbius group of analytic homeomorphisms of the unit disc in the complex plane onto itself. The elements of $\mathcal{M}$ have the form

$$\varphi(z) = \alpha(z-a)(1-\bar{a}z)^{-1} \quad (|\alpha| = 1, \ |a| < 1). \tag{1}$$

We assume that the spectrum of T is contained in the closed unit disc, so that $\varphi(T)$ is defined. (It is known that Möbius boundedness is equivalent to a first order growth condition on the resolvent; see Proposition 3.)

In this note we present a simple example of an operator that is Möbius bounded but not power bounded (in fact, $\|T^n\| = n+1$); this answers a question of B. M. Schreiber. We first present two propositions indicating the relationship between the two concepts.

In preparing this note the author benefitted from discussions with C. Foiaş and J. G. Stampfli.

Proposition 1. *If T is power bounded then it is Möbius bounded.*

Proof. Let $\varphi(z) = \sum \hat{\varphi}(n) z^n$ where φ is given by (1). One verifies that $\hat{\varphi}(0) = -\alpha a$, and $\hat{\varphi}(n) = \alpha(1-|a|^2)(\bar{a})^{n-1}$ $(n>0)$. Hence $\sum |\hat{\varphi}(n)| = 1 + 2|a| < 3$, and so $\|\varphi(T)\| < 3M$.

Proposition 2. *Let T be Möbius bounded with constant c, then*

$$\|T^n\| \leqq \frac{ce}{2}(n+1) \quad (n > 0).$$

Proof. $\varphi(T) = \sum \hat{\varphi}(n) T^n$. Hence

$$\hat{\varphi}(n) T^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(e^{i\theta} T) e^{-in\theta} \, d\theta,$$

Received August 8, 1977.

and so

$$\text{(2)} \qquad |\hat{\varphi}(n)|\,\|T^n\| \leqq \frac{1}{2\pi}\int \|\varphi(e^{i\theta}T)\|\,d\theta \leqq c.$$

Thus
$$\|T^n\| \leqq \frac{c}{(1-|a|^2)|a|^{n-1}} \quad (n>0).$$

For fixed n we let $a^2=1-2/(n+1)$. Then $\|T^n\|\leqq(n+1)c/2w_n$, where $w_n=$ $=[1-2/(n+1)]^{(n-1)/2}$. Since $(1-1/x)^{x-1}$ decreases to $1/e$ we see that w_n decreases to $1/e$, which completes the proof.

Inequality (2) shows that the proposition remains true under the weaker hypothesis that $\int\|\varphi(e^{i\theta}T)\|\,d\theta$ is bounded $(\varphi\in\mathcal{M})$. Thus one might hope that a better result could be obtained from the hypothesis that T is Möbius bounded. The following example shows that this is not the case.

Theorem 1. *There exists a Banach space X and an operator T such that $\|\varphi(T)\|=$ $=\|T\|=2$ $(\varphi\in\mathcal{M})$, and $\|T^n\|=n+1$ $(n\geqq 0)$.*

Proof. The elements of X are all those functions $f(z)$, analytic in the open unit disc, for which $f'\in H^1$. Geometrically this is equivalent to saying that f maps the unit disc onto a Riemann surface with a perimeter of finite length (see Duren [2], Theorem 3.12, for the case when f is a conformal map onto a plane domain bounded by a Jordan curve). By an inequality of G. H. Hardy ([2], Corollary to Theorem 3.15) each such function has an absolutely convergent power series, and hence is continuous on the closed disc. We norm X by taking the sum of the supremum norm of f and the H^1 norm of f':

$$\|f\| = \|f\|_\infty + \|f'\|_1.$$

One verifies that f is a commutative Banach algebra with identity under ordinary multiplication: $\|fg\|\leqq\|f\|\,\|g\|$, $\|1\|=1$.

The elements of X may be viewed as operators on X, operating by multiplication; the operator norm coincides with the norm in X.

For our operator T we take the operator M_z of multiplication by z. By the remark above: $\|T^n\|=\|z^n\|=n+1$ $(n\geqq 0)$.

Finally, we show that T is Möbius bounded. Let $\varphi\in\mathcal{M}$. Then one verifies that $\varphi(T)=M_\varphi$, the operator of multiplication by φ. Hence $\|\varphi(T)\|=\|\varphi\|$. Also, if φ is given by (1), then

$$\|\varphi'\|_1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|a|^2}{|1-\bar{a}e^{i\theta}|^2}\,d\Theta = 1.$$

(The integrand is the Poisson kernel.) Hence $\|\varphi\|=2$, which completes the proof.

Incidentally, it can be shown that the Möbius group operates on X by composition as a group of isometries:

$$\text{(3)} \qquad \|f\circ\varphi\| = \|f\| \quad (f\in X,\ \varphi\in\mathcal{M}).$$

This is obvious from the geometric interpretation of the elements of X, and it can be shown analytically by a change of variables in calculating $\|(f\circ\varphi)'\|_1$.

Question. If T is a Möbius bounded operator on Hilbert space do we have $\|T^n\|\leqq c(n+1)^{1/2}$?

We can prove this if $\|T^n\|$ is increasing, and if there is a unit vector f such that $\|T^n f\|\geqq\|T^n\|/2$ $(n>0)$. We omit the details. C. A. McCarthy proves this under a stronger hypothesis than Möbius boundedness (see Remark 3 at the end of this paper).

It is known that the condition of Möbius boundedness is equivalent to first order growth of the resolvent. We include a proof for completeness.

Proposition 3. *Let T be an operator with spectrum in the closed unit disc. Then T is Möbius bounded if and only if there is a constant d such that*

$$\|(T-\lambda)^{-1}\|\leqq\frac{d}{|\lambda|-1}\quad(1<|\lambda|<\infty). \tag{4}$$

Proof. Let $\lambda=1/a$ $(|a|<1)$. One verifies that (4) is equivalent to

$$\|(1-aT)^{-1}\|\leqq\frac{d}{1-|a|}\quad(|a|<1). \tag{5}$$

If φ is given by (1) then $1+\overline{a}\alpha\varphi(z)=(1-|a|^2)(1-\bar{a}z)^{-1}$. Thus

$$\|1+\overline{a}\alpha\varphi(T)\|=(1-|a|^2)\|(1-\bar{a}T)^{-1}\|. \tag{6}$$

Finally, (4) is equivalent to the boundedness of the right side of (6), while Möbius boundedness is equivalent to the boundedness of the left side of (6) (in showing that T is Möbius bounded it is sufficient to restrict attention to the parameter range $\frac{1}{2}<|a|<1$). This completes the proof.

We make a few remarks concerning the special case when T is an operator on Hilbert space.

Remark 1. B. Sz.-Nagy and C. Foiaş ([9], Remark 3, p. 20) have shown that if T satisfies (4) merely for $1<|\lambda|<1+\varepsilon$ for some $\varepsilon>0$, but with $d=1$, then T is in some class C_ϱ and hence is power bounded.

Remark 2. In [3] (Satz 4.1, p. 164) H.-O. Kreiss showed that if an operator on a finite-dimensional space satisfies (4), then it is power bounded (the bound depends on the dimension of the space). A shorter proof was given by K. W. Morton [7]. (This result was needed in studying the stability of finite-difference approximations to partial differential equations.)

On an infinite-dimensional space even the stronger assumption that the spectrum of T is a subset of the unit circumference and that (4) holds for all $|\lambda|\neq 1$ does not imply power boundedness. An example is given, somewhat implicitly, by C. A. McCarthy and J. T. Schwartz ([6], p. 199) (they state the growth condition (4) only for $|\lambda|>1$).

Closely related to this is an example due to A. S. MARCUS ([4], p. 544) of an operator A with real spectrum, that is not similar to a self-adjoint operator, but for which $\|(A-\lambda)^{-1}\| \leqq c|\mathrm{Im}\,\lambda|^{-1}$ ($\mathrm{Im}\,\lambda \neq 0$). (Such an example can also be obtained from the McCarthy—Schwartz example.)

In the positive direction, if (4) holds for all $|\lambda| \neq 1$, with $d=1$, then T is a unitary operator (see W. F. DONOGHUE [1]; see J. G. STAMPFLI [8], Theorem 2, for a generalization characterizing normal operators with spectrum contained in a smooth curve).

Remark 3. C. A. MCCARTHY [5] has considered the strong resolvent condition

$$\|(T-\lambda)^{-k}\| \leqq \frac{d}{(|\lambda|-1)^k} \quad (k = 1, 2, \ldots). \tag{7}$$

He shows that if T is an operator on a Banach space and if T satisfies (7), then $\|T^n\| \leqq 4n^{1/2}$ ($n=1, 2, \ldots$). Also, given $\varepsilon > 0$ he produces an example of an operator on Hilbert space that satisfies (7) with $d=1+\varepsilon$, but is not power bounded (the powers grow like $(\log\log n)^{1/2}$). Finally, he gives a more complicated example of an operator T whose spectrum is the unit circumference, such that both T and T^{-1} satisfy (7) with $d=1+\varepsilon$, but neither T nor T^{-1} is power bounded (again the powers grow like $(\log\log n)^{1/2}$).

Bibliography

[1] W. F. DONOGHUE, On a problem of Nieminen, *Inst. Hautes Etudes Sci., Publ. Math.*, **16** (1963), 31—33 (cumulative pagination: 127—129).

[2] P. L. DUREN, *Theory of H^p spaces*, Academic Press (New York, 1969).

[3] H.-O. KREISS, Über die Stabilitätsdefinition für Differenzengleichungen, die partielle Differentialgleichungen approximieren, *Nordisk Tidskrift for Informationsbehandling* (BIT), **2** (1962), 153—181.

[4] A. S. MARCUS, Some criteria for the completeness of the system of root vectors of a linear operator in a Banach space, *Matem. Sbornik*, **70** (112), No. 4 (1966), 526—561. (Russian)

[5] C. A. MCCARTHY, A strong resolvent condition does not imply power-boundedness, *Chalmers Institute of Technology and the University of Göteborg*, Preprint No. 15 (1971).

[6] C. A. MCCARTHY and J. T. SCHWARTZ, On the norm of a finite Boolean algebra of projections, and applications to theorems of Kreiss and Morton, *Commun. Pure and Applied Math.*, **18** (1965), 191—201.

[7] K. W. MORTON, On a matrix theorem due to H. O. Kreiss, *Commun. Pure and Applied Math.* **17** (1964), 375—379.

[8] J. G. STAMPFLI, A local spectral theory for operators, *J. Funct. Anal.*, 4 (1969), 1—10.

[9] B. SZ.-NAGY and C. FOIAŞ, On certain classes of power-bounded operators in Hilbert space, *Acta Sci. Math.*, **27** (1966), 17—25.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MICHIGAN
ANN ARBOR, MICHIGAN 48109